Bisection Method

"Essential Mathematica for Students of Science",James J. Kelly , 2006

http://www.physics.umd.edu/courses/CourseWare

"*Mathematica 4.1* Notebooks -**Complimentary software to accompany our textbook"**,John H. Mathews, and Russell W. Howell, 2002

Bisection Method

Background. The bisection method is one of the bracketing methods for finding roots of equations.

Implementation. Given a function f(x) and an interval which might contain a root, perform a predetermined number of iterations using the bisection method.

Limitations. Investigate the result of applying the bisection method over an interval where there is a discontinuity. Apply the bisection method for a function using an interval where there are distinct roots. Apply the bisection method over a "large" interval.

Theorem (Bisection Theorem). Assume that $f \in \mathbf{C}[a, b]$ and that there exists a number $\lambda \in [a, b]$ such that $f(\lambda) = 0$.

If $f(a)$ and $f(b)$ have opposite signs, and $\{c_n\}$ represents the sequence of midpoints generated by the bisection process, then

$$\left| \lambda - c_n \right| \le \frac{b-a}{2^{n+1}} \quad \text{for} \quad n = 0, 1, \ldots,$$

and the sequence $\{c_n\}$ converges to the zero $x = \lambda$.

That is, $\lim_{k \to \infty} c_n = \lambda$.

```
Bisection[a0_, b0_, λ_] :=
 Module[{},
  a = N[a0];
  b = N[b0];
  c = (a + b)/2;
  i = 0;
  output = {{i, a, c, b, f[c]}};
  While[i < λ,
   If[ Sign[f[b]] == Sign[f[c]],
    b = c, a = c; ];
   c = (a + b)/2;
   i = i + 1;
   output = Append[output, {i, a, c, b, f[c]}]; ];
  Print[NumberForm[TableForm[output,
     TableHeadings → {None, {"i", "ak", "ck", "bk", "f[ck]"}}], 16]];
  Print["  c   = ", NumberForm[c, 16] ];
  Print[" Δc   = ±", (b - a)/2];
  Print["f[c]  = ", NumberForm[f[c], 16] ]; ]
```

Example 1. Find all the real solutions to the cubic equation $x^3 + 2x^2 - 5 = 0$.

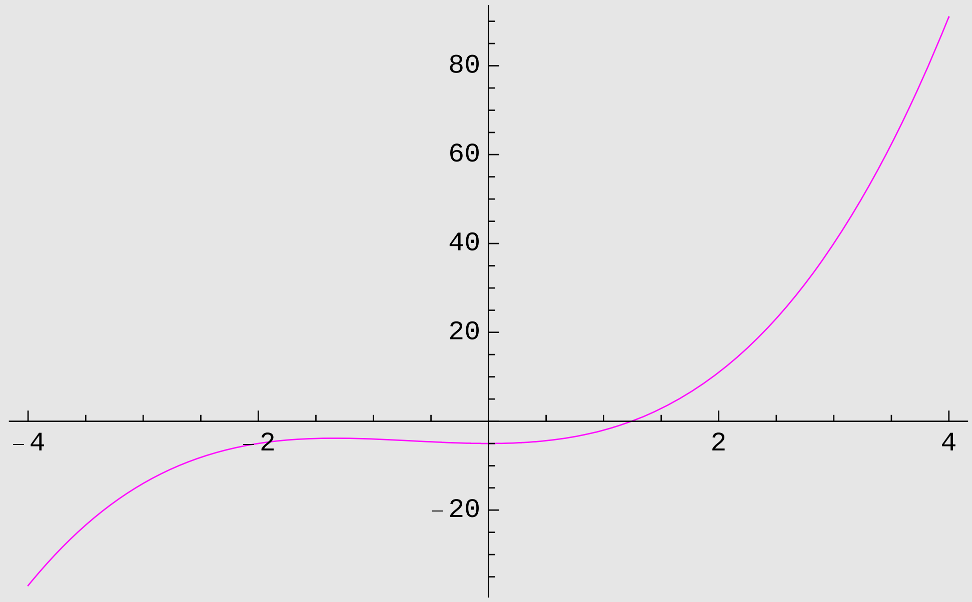
Solution

```
f[x_] = x^3 + 2 x^2 - 5;
Print["f[x]  = ", f[x] ];
```

$f[x] = -5 + 2x^2 + x^3$

Plot the function.

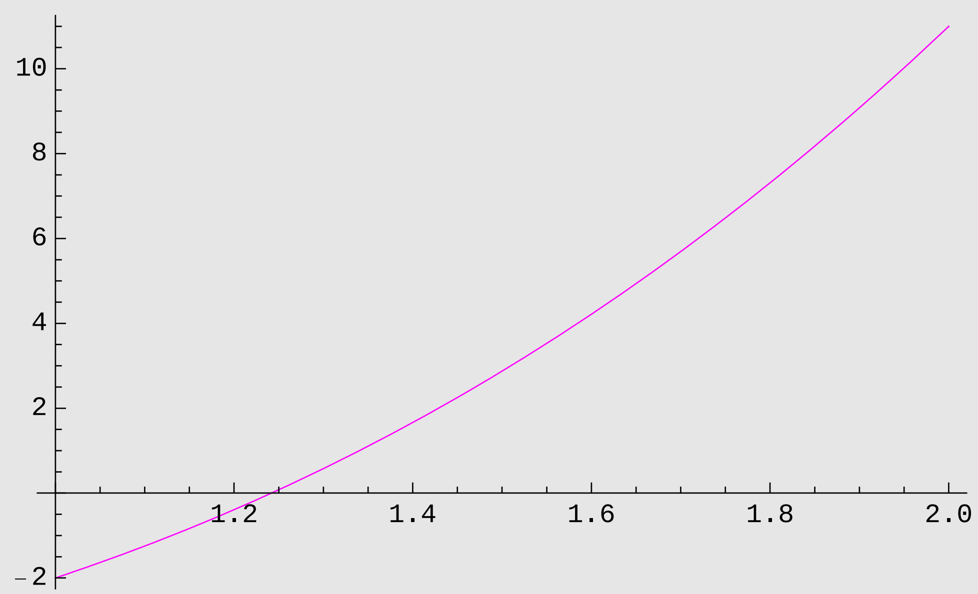
```
Needs["Graphics`Colors`"]
Plot[f[x], {x, -4, 4}, PlotStyle → Magenta]
Print["y = f[x] = ", f[x]];
```



y = f[x] = $-5 + 2x^2 + x^3$

There appears to be only one real root which lies in the interval [1,2].

```
Plot[f[x], {x, 1, 2}, PlotStyle → Magenta]
Print["y = f[x] = ", f[x]];
```



y = f[x] = $-5+2x^2+x^3$

Call the Bisection subroutine on the interval [1,2] using 10 iterations

```
Bisection[1, 2, 10];
```

i	a_k	c_k	b_k	$f[c_k]$
0	1.	1.5	2.	2.875
1	1.	1.25	1.5	0.078125
2	1.	1.125	1.25	-1.044921875
3	1.125	1.1875	1.25	-0.505126953125
4	1.1875	1.21875	1.25	-0.219024658203125
5	1.21875	1.234375	1.25	-0.07184219360351562
6	1.234375	1.2421875	1.25	0.002791881561279297
7	1.234375	1.23828125	1.2421875	-0.03461235761642456
8	1.23828125	1.240234375	1.2421875	-0.01593206077814102
9	1.240234375	1.2412109375	1.2421875	-0.006575548090040684
10	1.2412109375	1.24169921875	1.2421875	-0.001893198234029114

c = 1.24169921875

△c = ±0.000488281

f[c] = -0.001893198234029114

After 10 iterations, the interval has been reduced to [a,b] where

```
Print["a = ", NumberForm[a, 16] ];
Print["b = ", NumberForm[b, 16] ];
Print[""];
Print["[a, b] = [", a, ", ", b, "]"];
```

```
a = 1.2412109375

b = 1.2421875


[a, b] = [1.24121, 1.24219]
```

The root lies somewhere in the interval [a,b] the width of which is

```
Print["b-a = ", NumberForm[b - a, 16] ];
```

```
b-a = 0.0009765625
```

The reported root is alleged to be

```
Print["c = ", NumberForm[c, 16] ];
```

```
c = 1.24169921875
```

The accuracy we can guarantee is one half of the interval width.

```
Print["(b - a)/2 = ", NumberForm[(b - a)/2, 16] ];
```

$\frac{b-a}{2}$ = 0.00048828125

Is this the desired accuracy you want ? If not, more iterations are required.

```
Bisection[1, 2, 20];
```

i	a_k	c_k	b_k	$f[c_k]$
0	1.	1.5	2.	2.875
1	1.	1.25	1.5	0.078125
2	1.	1.125	1.25	-1.044921875
3	1.125	1.1875	1.25	-0.505126953125
4	1.1875	1.21875	1.25	-0.21902465820312
5	1.21875	1.234375	1.25	-0.07184219360351
6	1.234375	1.2421875	1.25	0.00279188156127
7	1.234375	1.23828125	1.2421875	-0.03461235761642
8	1.23828125	1.240234375	1.2421875	-0.01593206077814
9	1.240234375	1.2412109375	1.2421875	-0.00657554809004
10	1.2412109375	1.24169921875	1.2421875	-0.00189319823402
11	1.24169921875	1.241943359375	1.2421875	0.00044900037755
12	1.24169921875	1.2418212890625	1.241943359375	-0.00072218424429
13	1.2418212890625	1.24188232421875	1.241943359375	-0.00013661326306
14	1.24188232421875	1.241912841796875	1.241943359375	0.00015618822473
15	1.24188232421875	1.241897583007812	1.241912841796875	9.7861477179606 × 1
16	1.24188232421875	1.241889953613281	1.241897583007812	-0.00006341389095
17	1.241889953613281	1.241893768310547	1.241897583007812	-0.00002681395493
18	1.241893768310547	1.24189567565918	1.241897583007812	-8.51392443923515
19	1.24189567565918	1.241896629333496	1.241897583007812	6.36106431528560
20	1.24189567565918	1.241896152496338	1.241896629333496	-3.93891030547877

c = 1.241896152496338

$\triangle c = \pm 4.76837 \times 10^{-7}$

$f[c] = -3.938910305478771 \times 10^{-6}$

Compare our result with Mathematica's built in root finder.

```
sols = NSolve[f[x] == 0, x];
NumberForm[TableForm[sols], 16]
```

```
x → -1.62094828151724 - 1.182635560091155 i
x → -1.62094828151724 + 1.182635560091155 i
x → 1.24189656303448
```

Question. Why is *Mathematica*'s answer different ?

How many bisections would it take to reduce the interval width to 10^{-16} ?

```
Bisection[1, 2, 50];
```

i	a_k	c_k	b_k	$f[c_k]$
0	1.	1.5	2.	2.875
1	1.	1.25	1.5	0.078125
2	1.	1.125	1.25	−1.044921875
3	1.125	1.1875	1.25	−0.505126953125
4	1.1875	1.21875	1.25	−0.21902465820312
5	1.21875	1.234375	1.25	−0.07184219360351
6	1.234375	1.2421875	1.25	0.002791881561279
7	1.234375	1.23828125	1.2421875	−0.03461235761642
8	1.23828125	1.240234375	1.2421875	−0.01593206077814
9	1.240234375	1.2412109375	1.2421875	−0.00657554809004
10	1.2412109375	1.24169921875	1.2421875	−0.00189319823402
11	1.24169921875	1.241943359375	1.2421875	0.000449000377557
12	1.24169921875	1.2418212890625	1.241943359375	−0.00072218424429
13	1.2418212890625	1.24188232421875	1.241943359375	−0.00013661326306
14	1.24188232421875	1.241912841796875	1.241943359375	0.000156188224735
15	1.24188232421875	1.241897583007812	1.241912841796875	9.7861477179606 ×
16	1.24188232421875	1.241889953613281	1.241897583007812	−0.00006341389095
17	1.241889953613281	1.241893768310547	1.241897583007812	−0.00002681395493
18	1.241893768310547	1.24189567565918	1.241897583007812	−8.51392443923515
19	1.24189567565918	1.241896629333496	1.241897583007812	6.361064315285603
20	1.24189567565918	1.241896152496338	1.241896629333496	−3.93891030547877
21	1.241896152496338	1.241896390914917	1.241896629333496	−1.65140226204840
22	1.241896390914917	1.241896510124207	1.241896629333496	−5.07647996528248
23	1.241896510124207	1.241896569728851	1.241896629333496	6.42291970720521 ›
24	1.241896510124207	1.241896539926529	1.241896569728851	−2.21709404613079
25	1.241896539926529	1.24189655482769	1.241896569728851	−7.87401051027814
26	1.24189655482769	1.241896562278271	1.241896569728851	−7.25545401536464
27	1.241896562278271	1.241896566003561	1.241896569728851	2.848687152834373
28	1.241896562278271	1.241896564140916	1.241896566003561	1.061570831240033
29	1.241896562278271	1.241896563209593	1.241896564140916	1.68012714851784 ›
30	1.241896562278271	1.241896562743932	1.241896563209593	−2.78766343342340
31	1.241896562743932	1.241896562976763	1.241896563209593	−5.53768586541991
32	1.241896562976763	1.241896563093178	1.241896563209593	5.631797250771342
33	1.241896562976763	1.24189656303497	1.241896563093178	4.705569267571263
34	1.241896562976763	1.241896563005866	1.24189656303497	−2.74530620458790
35	1.241896563005866	1.241896563020418	1.24189656303497	−1.34912525595609
36	1.241896563020418	1.241896563027694	1.24189656303497	−6.51034781640191
37	1.241896563027694	1.241896563031332	1.24189656303497	−3.01989544482239
38	1.241896563031332	1.241896563033151	1.24189656303497	−1.27471366795362
39	1.241896563033151	1.241896563034061	1.24189656303497	−4.01989552756276
40	1.241896563034061	1.241896563034516	1.24189656303497	3.428368700042483
41	1.241896563034061	1.241896563034288	1.241896563034516	−1.83852932877925
42	1.241896563034288	1.241896563034402	1.241896563034516	−7.47846229387505
43	1.241896563034402	1.241896563034459	1.241896563034516	−2.02504679691628

44	1.241896563034459	1.241896563034487	1.241896563034516	7.016609515630989
45	1.241896563034459	1.241896563034473	1.241896563034487	−6.57252030578092
46	1.241896563034473	1.24189656303448	1.241896563034487	1.77635683940025›
47	1.241896563034473	1.241896563034476	1.24189656303448	−3.19744231092045
48	1.241896563034476	1.241896563034478	1.24189656303448	−1.50990331349021
49	1.241896563034478	1.241896563034479	1.24189656303448	−7.10542735760100
50	1.241896563034479	1.24189656303448	1.24189656303448	−2.66453525910037

c = 1.24189656303448

△c = $\pm 4.44089 \times 10^{-16}$

f[c] = $-2.664535259100376 \times 10^{-15}$

Remember. The bisection method can only be used to find a real root in an interval [a,b] in which f[x] changes sign.

Example 2. Use the cubic equation $x^3 + 2x^2 - 5 = 0$ in Example 1 and perform the following call to the bisection method.

Bisec-tion[0,1,30];

Solution

```
Bisection[0, 1, 30];
```

i	a_k	c_k	b_k	$f[c_k]$
0	0.	0.5	1.	-4.375
1	0.	0.25	0.5	-4.859375
2	0.	0.125	0.25	-4.966796875
3	0.	0.0625	0.125	-4.991943359375
4	0.	0.03125	0.0625	-4.998016357421875
5	0.	0.015625	0.03125	-4.999507904052734
6	0.	0.0078125	0.015625	-4.999877452850342
7	0.	0.00390625	0.0078125	-4.99996942281723
8	0.	0.001953125	0.00390625	-4.999992363154888
9	0.	0.0009765625	0.001953125	-4.999998091720045
10	0.	0.00048828125	0.0009765625	-4.999999523046426
11	0.	0.000244140625	0.00048828125	-4.999999880776159
12	0.	0.0001220703125	0.000244140625	-4.999999970195859
13	0.	0.00006103515625	0.0001220703125	-4.999999992549192
14	0.	0.000030517578125	0.00006103515625	-4.999999998137326
15	0.	0.0000152587890625	0.000030517578125	-4.999999999534335
16	0.	$7.62939453125\times10^{-6}$	0.0000152587890625	-4.999999999883585
17	0.	$3.814697265625\times10^{-6}$	$7.62939453125\times10^{-6}$	-4.999999999970896
18	0.	$1.9073486328125\times10^{-6}$	$3.814697265625\times10^{-6}$	-4.999999999992724
19	0.	$9.5367431640625\times10^{-7}$	$1.9073486328125\times10^{-6}$	-4.999999999998181
20	0.	$4.76837158203125\times10^{-7}$	$9.5367431640625\times10^{-7}$	-4.999999999999545
21	0.	$2.384185791015625\times10^{-7}$	$4.76837158203125\times10^{-7}$	-4.999999999999886
22	0.	$1.192092895507812\times10^{-7}$	$2.384185791015625\times10^{-7}$	-4.999999999999972
23	0.	$5.960464477539062\times10^{-8}$	$1.192092895507812\times10^{-7}$	-4.999999999999993
24	0.	$2.980232238769531\times10^{-8}$	$5.960464477539062\times10^{-8}$	-4.999999999999998
25	0.	$1.490116119384766\times10^{-8}$	$2.980232238769531\times10^{-8}$	-5.
26	0.	$7.450580596923828\times10^{-9}$	$1.490116119384766\times10^{-8}$	-5.
27	0.	$3.725290298461914\times10^{-9}$	$7.450580596923828\times10^{-9}$	-5.
28	0.	$1.862645149230957\times10^{-9}$	$3.725290298461914\times10^{-9}$	-5.
29	0.	$9.31322574615479\times10^{-10}$	$1.862645149230957\times10^{-9}$	-5.
30	0.	$4.656612873077393\times10^{-10}$	$9.31322574615479\times10^{-10}$	-5.

$c = 4.656612873077393\times10^{-10}$

$\triangle c = \pm4.65661\times10^{-10}$

$f[c] = -5.$

Caution. You must be able to explain what the output means.

Reduce the volume of printout.

After you have debugged you program and it is working properly, delete the unnecessary print statements.

Concise Program for the Bisection Method

```
Bisection[a0_, b0_, λ_] :=
  Module[{a = N[a0], b = N[b0]},
   c = (a + b)/2;
   i = 0;
   While[i < λ,
    If[ Sign[f[b]] == Sign[f[c]],
     b = c, a = c; ];
    c = (a + b)/2;
    i = i + 1; ];
   Print["  c   = ", NumberForm[c, 16] ];
   Print[" Δc   = ±", (b - a)/2];
   Print["f[c]  = ", NumberForm[f[c], 16] ]; ];
```

Now test the example to see if it still works. Use the last case in Example 1 given above and compare with the previous results.

```
Bisection[1, 2, 30];
```

  c   = 1.241896562743932

 Δc   = $\pm 4.65661 \times 10^{-10}$

f[c] = $-2.787663433423404 \times 10^{-9}$

Reducing the Computational Load for the Bisection Method

The following program uses fewer computations in the bisection method and is the traditional way to do it. Can you determine how many fewer functional evaluations are used ?

```
Bisection[a0_, b0_, λ_] :=
Module[{a = N[a0], b = N[b0]},
  c = (a + b)/2;
  Yb = f[b];
  Yc = f[c];
  i = 0;
  While[i < λ,
   If[ Sign[Yb] == Sign[Yc],
    b = c;
    Yb = Yc,
    a = c; ];
   c = (a + b)/2;
   Yc = f[c];
   i = i + 1; ];
 Print["  c   = ", NumberForm[c, 16] ];
 Print[" Δc   = ±", (b - a)/2];
 Print["f[c]  = ", NumberForm[Yc, 16] ]; ]
```

```
Bisection[1, 2, 30];
```

  c = 1.241896562743932

 Δc = $\pm 4.65661 \times 10^{-10}$

f[c] = $-2.787663433423404 \times 10^{-9}$

Various Scenarios and Animations for the Bisection Method.

```
Bisection[a0_, b0_, λ_] :=
 Module[{},
  a = N[a0];
  b = N[b0];
  c = (a + b)/2;
  i = 0;
  output = {{i, a, c, b, f[c]}};
  While[i < λ,
   If[ Sign[f[b]] == Sign[f[c]],
    b = c, a = c; ];
   c = (a + b)/2;
   i = i + 1;
   output = Append[output, {i, a, c, b, f[c]}]; ];
  Print[NumberForm[TableForm[output,
     TableHeadings → {None, {"k", "a_k", "c_k", "b_k", "f[c_k]"}}], 16]];
  Print["  c  = ", NumberForm[c, 16] ];
  Print[" Δc  = ±", (b - a)/2];
  Print["f[c] = ", NumberForm[f[c], 16] ]; ]
```

```
Bisection[1, 2, 30];
```

k	a_k	c_k	b_k	$f[c_k]$
0	1.	1.5	2.	2.875
1	1.	1.25	1.5	0.078125
2	1.	1.125	1.25	−1.044921875
3	1.125	1.1875	1.25	−0.505126953125
4	1.1875	1.21875	1.25	−0.21902465820312
5	1.21875	1.234375	1.25	−0.07184219360351
6	1.234375	1.2421875	1.25	0.002791881561279
7	1.234375	1.23828125	1.2421875	−0.03461235761642
8	1.23828125	1.240234375	1.2421875	−0.01593206077814
9	1.240234375	1.2412109375	1.2421875	−0.00657554809004
10	1.2412109375	1.24169921875	1.2421875	−0.00189319823402
11	1.24169921875	1.241943359375	1.2421875	0.000449000377557
12	1.24169921875	1.2418212890625	1.241943359375	−0.00072218424429
13	1.2418212890625	1.24188232421875	1.241943359375	−0.00013661326306
14	1.24188232421875	1.241912841796875	1.241943359375	0.000156188224735
15	1.24188232421875	1.241897583007812	1.241912841796875	9.7861477179606 × 1
16	1.24188232421875	1.241889953613281	1.241897583007812	−0.00006341389095
17	1.241889953613281	1.241893768310547	1.241897583007812	−0.00002681395493
18	1.241893768310547	1.24189567565918	1.241897583007812	−8.51392443923515
19	1.24189567565918	1.241896629333496	1.241897583007812	6.361064315285603
20	1.24189567565918	1.241896152496338	1.241896629333496	−3.93891030547877
21	1.241896152496338	1.241896390914917	1.241896629333496	−1.65140226204840
22	1.241896390914917	1.241896510124207	1.241896629333496	−5.07647996528248
23	1.241896510124207	1.241896569728851	1.241896629333496	6.42291970720521 ›
24	1.241896510124207	1.241896539926529	1.241896569728851	−2.21709404613079
25	1.241896539926529	1.24189655482769	1.241896569728851	−7.87401051027814
26	1.24189655482769	1.241896562278271	1.241896569728851	−7.25545401536464
27	1.241896562278271	1.241896566003561	1.241896569728851	2.848687152834373
28	1.241896562278271	1.241896564140916	1.241896566003561	1.061570831240033
29	1.241896562278271	1.241896563209593	1.241896564140916	1.68012714851784 ›
30	1.241896562278271	1.241896562743932	1.241896563209593	−2.78766343342340

$c = 1.241896562743932$

$\Delta c = \pm 4.65661 \times 10^{-10}$

$f[c] = -2.787663433423404 \times 10^{-9}$

Example 3. Convergence Find the solution to the cubic equation $x^3 + 2x^2 - 5 = 0$. Use the starting interval $[a, b] = [-1, 2]$.

Solution

```
Needs["Graphics`Colors`"];
(f[x_] = x^3 + 2 x^2 - 5;) (a = -1.`;) (b = 2.`;)
Bisection[a, b, 30];
 Plot[f[x], {x, -1.05`, 2.05`}, PlotRange → {{-1.05`, 2.05`}, {-10, 15}},
  PlotStyle → Magenta]
```

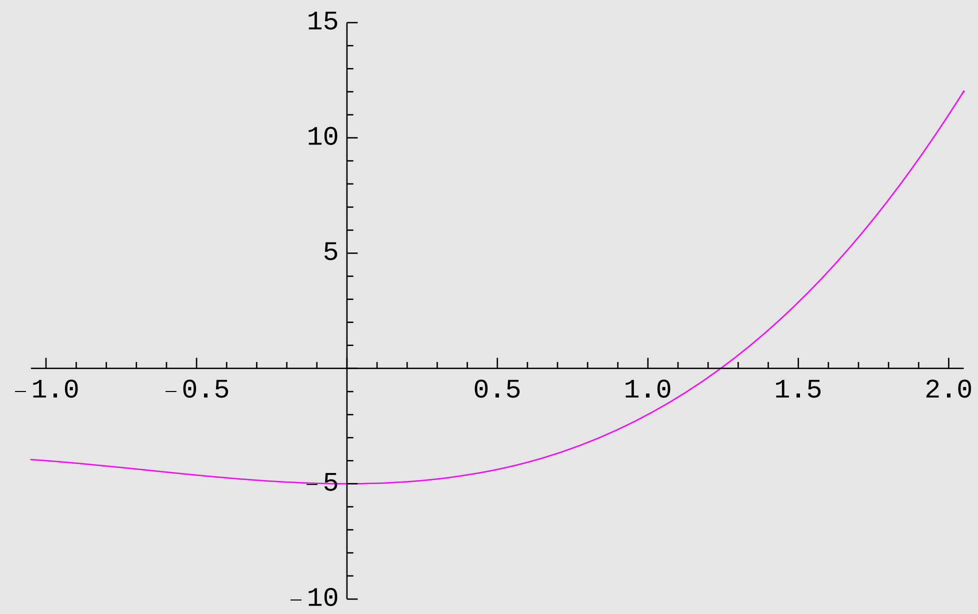
```
Null^3
```

k	a_k	c_k	b_k	$f[c_k]$
0	−1.	0.5	2.	−4.375
1	0.5	1.25	2.	0.078125
2	0.5	0.875	1.25	−2.798828125
3	0.875	1.0625	1.25	−1.542724609375
4	1.0625	1.15625	1.25	−0.78036499023437
5	1.15625	1.203125	1.25	−0.36344528198242
6	1.203125	1.2265625	1.25	−0.14578008651733
7	1.2265625	1.23828125	1.25	−0.03461235761642
8	1.23828125	1.244140625	1.25	0.021559514105319
9	1.23828125	1.2412109375	1.244140625	−0.00657554809004
10	1.2412109375	1.24267578125	1.244140625	0.007479691994376
11	1.2412109375	1.241943359375	1.24267578125	0.000449000377557
12	1.2412109375	1.2415771484375	1.241943359375	−0.00306404160255
13	1.2415771484375	1.24176025390625	1.241943359375	−0.00130771256749
14	1.24176025390625	1.241851806640625	1.241943359375	−0.00042940408602
15	1.241851806640625	1.241897583007812	1.241943359375	9.7861477179606 ×
16	1.241851806640625	1.241874694824219	1.241897583007812	−0.00020981196862
17	1.241874694824219	1.241886138916016	1.241897583007812	−0.00010001366032
18	1.241886138916016	1.241891860961914	1.241897583007812	−0.00004511394377
19	1.241891860961914	1.241894721984863	1.241897583007812	−0.00001766394489
20	1.241894721984863	1.241896152496338	1.241897583007812	−3.93891030547877
21	1.241896152496338	1.241896867752075	1.241897583007812	2.923615777028488
22	1.241896152496338	1.241896510124207	1.241896867752075	−5.07647996528248
23	1.241896510124207	1.241896688938141	1.241896867752075	1.207983706841276
24	1.241896510124207	1.241896599531174	1.241896688938141	3.501678094153249
25	1.241896510124207	1.24189655482769	1.241896599531174	−7.87401051027814
26	1.24189655482769	1.241896577179432	1.241896599531174	1.357138490476473
27	1.24189655482769	1.241896566003561	1.241896577179432	2.848687152834373
28	1.24189655482769	1.241896560415626	1.241896566003561	−2.51266172313080
29	1.241896560415626	1.241896563209593	1.241896566003561	1.68012714851784 ›
30	1.241896560415626	1.241896561812609	1.241896563209593	−1.17232445973058

c = 1.241896561812609

$\triangle c = \pm 1.39698 \times 10^{-9}$

$f[c] = -1.172324459730589 \times 10^{-8}$


15
10
5
-1.0
-0.5
0.5
1.0
1.5
2.0
-5
-10